THE RANGE OF HOLOMORPHIC MAPS AT BOUNDARY POINTS

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Abstract. We prove a boundary version of the open mapping theorem for holomorphic maps between strongly pseudoconvex domains. That is, we prove that the local image of a holomorphic map $f : D \to D'$ close to a boundary regular contact point $p \in \partial D$ where the Jacobian is bounded from zero along normal non-tangential directions has to eventually contain every cone (and more generally every admissible region) with vertex at $f(p)$.

1. Introduction

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. If $\lim_{(0,1)\ni r\to 1} f(r) = 1$ and $L := \liminf_{\zeta\to 1} \frac{1-|f(\zeta)|}{1-|\zeta|} < +\infty$, the point 1 is called a *boundary regular fixed point* for f and, thanks to the classical Julia-Wolff-Carathéodory theorem (see, *e.g.* [1]), it follows that f has non-tangential limit 1 at 1 and $f'(\zeta)$ has non-tangential limit L at 1. In particular, f is isogonal at 1 and hence it maps angles in $\mathbb{D}$ with vertex at 1 into angles with vertex 1 and equal amplitude. Such a result has an interesting quantitative interpretation: every angle with vertex at 1 is eventually contained in the local image of f at 1. This can be considered a boundary version of the open mapping theorem, and it is the best one can say about the range of one-dimensional mappings close to boundary points.

In higher dimension, W. Rudin [9] for the unit ball and M. Abate [1, 2, 3] for strongly (pseudo)convex domains generalized from a qualitative point of view the classical Julia-Wolff-Carathéodory theorem. Such a theorem can be seen as a description of the possible one-jets for holomorphic mappings from a strongly pseudoconvex domain into another, close to a boundary point which is non-tangentially mapped to another boundary point. In this optic, in [6] a full description of all jets of such mappings, given some smooth extension, is provided. However, the question on how big the local image of the map close to the boundary point really must be, is not answered from the higher version of the classical Julia-Wolff-Carathéodory theorem. The aim of the present paper is precisely to give an answer to such a question.

In order to state our result, we need to introduce some notations (see Sections 2 and 3 for details). Let $D, D' \subset \mathbb{C}^n$ be two bounded strongly pseudoconvex domains with smooth

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boundary. Let $p \in \partial D$ and let $f : D \to D'$ be holomorphic. Assume

$$\liminf_{z \to p} \frac{\mathrm{dist}(f(z), \partial D')}{\mathrm{dist}(z, \partial D)} < +\infty.$$

Such a condition is natural and it is precisely the analogous of the one assumed in the classical Julia-Wolff-Carathéodory theorem. The point p is then called a *regular contact point* and Abate's version of the classical Julia-Wolff-Carathéodory theorem implies that there exists $q \in \partial D'$ such that f has non-tangential limit q at p. Without other assumption, the local image of f close to p can be very thin (although in Corollary 2.6 we prove that, for $D = D' = \mathbb{B}^n$ the unit ball of $\mathbb{C}^n$, the local range of f at p has to be Kobayashi asymptotic to every cone with vertex at q). This is not very surprising, since the same happens for points inside D whenever the Jacobian of f is zero.

We say that the point p is a *super-regular contact point* provided $\det df_z$ is bounded from zero when z tends to p non-tangentially along the normal direction to ∂D at p (see Definition 3.1). In particular, if f is of class C^1 at p, the point p is super-regular provided $\det df_p \neq 0$.

Let $\mathbb{B}(x, R)$ denote the Euclidean ball of center $x \in \mathbb{C}^n$ and radius $R > 0$. Our main result is the following:

Theorem 1.1. *Let $D, D' \subset \mathbb{C}^n$ be two bounded strongly pseudoconvex domains with smooth boundary. Let $f : D \to D'$ be holomorphic. Assume $p \in \partial D$ is a super-regular contact point for f. Then there exists a point $q \in \partial D'$ such that for every $\eta > 0$ and for every cone $C \subset D'$ with vertex at q there exists $\delta > 0$ such that $C \cap \mathbb{B}(q, \delta) \subset f(\mathbb{B}(p, \eta) \cap D)$.*

The conclusion of Theorem 1.1 holds more generally for the so-called *admissible* sets, namely, for those sets in D' which are asymptotic to cones in the Kobayashi distance (see, Section 3 and Theorem 3.6).

The proof of Theorem 1.1 is based on the corresponding theorem for the unit ball. In Section 2 we study the local image of holomorphic self-maps of the unit ball close to a regular and super-regular fixed point. In particular, the key result is Theorem 2.11, a sort of boundary Köbe 1/4-theorem, where we prove that the image close to a super-regular fixed point must contain all the Kobayashi balls of a fixed radius which are centered at points of the image of an angle in the normal directions. With such a result at hands, we get Theorem 1.1 (and its general version for admissible sets Theorem 3.6) by suitably embedding strongly pseudoconvex domains into the unit ball, using a recent result by the second named author with E. F. Wold and K. Diederich, see Section 3.

As an application of our result, in Theorem 4.1, we prove that if $f : D \to D'$ is univalent and $x \in \partial D$ is a super-regular contact point for f, then for every regular contact point $y \in \partial D \setminus \{x\}$ it holds $f(x) \neq f(y)$. Contrarily to the one-dimensional case (where regular and super-regular points coincide), this is the best one can say. In fact, in Example 4.2 we construct a univalent map of the unit ball having $(\pm 1, 0, \ldots, 0)$ as regular contact points (but not super-regular) such that $f(1, 0, \ldots, 0) = f(-1, 0, \ldots, 0)$.

2. The unit ball

For a point $a \in \mathbb{C}^n$ and $r > 0$ let denote by $\mathbb{B}(a,r) := \{z \in \mathbb{C}^n : \|z-a\| < r\}$. As customary, we let $\mathbb{B}^n := \mathbb{B}(0,1)$ and denote by $e_1 = (1,0,\dots,0)$.

Let $\pi : \mathbb{C}^n \to \mathbb{C}^n$ be defined as $\pi(z) = \pi(z_1,\dots,z_n) = (z_1,0,\dots,0)$.

Let $k_{\mathbb{B}^n}$ denote the Kobayashi distance in $\mathbb{B}^n$. Recall that

$$k_{\mathbb{B}^n}(a,b) = \frac{1}{2}\log\frac{1+\|T_a(b)\|}{1-\|T_a(b)\|},$$

where $T_a : \mathbb{B}^n \to \mathbb{B}^n$ is any automorphism such that $T_a(a) = 0$. We will use such an explicit formula in case $b = \pi(a)$. A direct computation from the explicit form of the automorphisms of $\mathbb{B}^n$ (see, *e.g.*, [1, p. 358] or below) gives

$$\|T_a(\pi(a))\|^2 = \frac{\|a-\pi(a)\|^2}{1-\|\pi(a)\|^2}. \tag{2.1}$$

For a subset $A \subset \mathbb{B}^n$ and $z \in \mathbb{B}^n$ we let $k_{\mathbb{B}^n}(z,A) = \inf_{w\in A} k_{\mathbb{B}^n}(z,w)$.

Also, for $z \in \mathbb{B}^n$ and $R > 0$ we let $B_k(z,R)$ denote the *Kobayashi ball* of center z and radius R.

Definition 2.1. Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. The point $e_1 = (1,0,\dots,0)$ is said to be a *boundary regular fixed point* of f if

(1) $\alpha_f(e_1) := \liminf_{z\to e_1} \frac{1-\|f(z)\|}{1-\|z\|} < +\infty$,
(2) $\lim_{(0,1)\ni r\to 1} f(re_1) = e_1$.

Let $R \geq 1$. The set $\{z \in \mathbb{B}^n : |1-z_1| \leq R(1-\|z\|)\}$ is a *Korányi region of vertex* e_1 *and amplitude* R (see [1, Section 2.2.3]). In [9, Section 5.4.1] a slightly different but essentially equivalent definition is given and used. In order not to excessively burden the notation, since we are only working at e_1, from now on, when we talk about Korányi regions, we will always mean Korányi regions of vertex e_1.

Let $f : \mathbb{B}^n \to \mathbb{C}^m$ be a holomorphic map. We say that f has *K-limit* L at e_1 – and we write $K\text{-}\lim_{z\to e_1} f(z) = L$ – if for each sequence $\{z_k\} \subset \mathbb{B}^n$ converging to e_1 such that $\{z_k\}$ belongs eventually to some Korányi region, it follows that $f(z_k) \to L$.

Let $M > 1$. We denote by $C(M) := \{z \in \mathbb{B}^n : \|e_1 - z\| < M(1-\|z\|)\}$ a *cone of vertex* e_1 *and amplitude* M. Also, let $M > 1, s \in (0,1)$. We say that f has *non-tangential limit* L at e_1 and we write $\angle\lim_{z\to e_1} f(z) = L$, if for each sequence $\{z_k\} \subset \mathbb{B}^n$ which is eventually contained in a cone of vertex e_1 and amplitude $M > 1$, it follows that $f(z_k) \to L$.

In dimension one, Korányi regions and cones are one and the same and in fact, studying boundary behavior of holomorphic mappings in the unit disc, it is natural to consider non-tangential limits. However in higher dimension cones are contained in Korányi regions, but the first are "too small" and the latter "too big" and one is forced to consider intermediate sets, which can be tangent to the unit ball in complex tangent directions but are "asymptotic" in hyperbolic terms.

Definition 2.2. Let $A \subset \mathbb{B}^n$ be such that $e_1 \in \overline{A}$. We say that A is *admissible* at e_1 if for every $\epsilon > 0$ there exists $\delta > 0$ and $M > 1$ such that

(1) $\pi(A \cap \mathbb{B}(e_1, \delta)) \subset C(M)$
(2) $k_{\mathbb{B}^n}(z, \pi(z)) < \epsilon$ for every $z \in A \cap \mathbb{B}(e_1, \delta)$.

We say that f has *restricted K-limit* (or *admissible limit*) L at e_1 – and we write $\angle_K \lim_{z \to e_1} f(z) = L$ – if for each sequence $\{z_k\} \subset \mathbb{B}^n$ converging to e_1 such that $\{z_k\}$ is admissible at e_1 it follows that $f(z_k) \to L$.

Note that if a sequence $\{z_k\} \subset \mathbb{B}^n$ converging to e_1 is admissible at e_1, then $\langle z_k, e_1 \rangle \to 1$ non-tangentially in $\mathbb{D}$ and

$$\lim_{k \to \infty} k_{\mathbb{B}^n}(z_k, \pi(z_k)) = 0.$$

By (2.1), this latter condition is equivalent to

$$\frac{\|z_k - \langle z_k, e_1 \rangle e_1\|^2}{1 - |\langle z_k, e_1 \rangle|^2} \to 0.$$

One can show that

$$K\text{-}\lim_{z \to e_1} f(z) = L \Longrightarrow \angle_K \lim_{z \to e_1} f(z) = L \Longrightarrow \angle \lim_{z \to e_1} f(z) = L,$$

but the converse to any of these implications is not true in general.

Lemma 2.3. *Every cone $C(M)$ in $\mathbb{B}^n$ with vertex e_1 and amplitude $M > 1$ is admissible at e_1.*

Proof. Let $C(M)$ be a cone with vertex e_1 and amplitude $M > 1$. Let $\epsilon > 0$. We want to prove that there exists $\delta > 0$ such that

$$k_{\mathbb{B}^n}(z, \pi(z)) < \epsilon \quad \forall z \in C(M) \cap \mathbb{B}(e_1, \delta).$$

By (2.1), this is equivalent to prove that for each $\eta > 0$ there exists $\delta > 0$ such that

$$\frac{\|z - z_1 e_1\|^2}{1 - |z_1|^2} < \eta \quad \forall z \in C(M) \cap \mathbb{B}(e_1, \delta). \tag{2.2}$$

But,

$$\frac{\|z - z_1 e_1\|^2}{1 - |z_1|^2} = \frac{\|z - z_1 e_1\|^2}{\|z - e_1\|^2} \frac{\|e_1 - z\|}{1 - |z_1|} \frac{\|z - e_1\|}{1 + |z_1|} \leq M \frac{1 - \|z\|}{1 - |z_1|} \|e_1 - z\| \leq M \|e_1 - z\|,$$

and therefore, if δ is sufficiently small, (2.2) follows. □

The following result is due to Rudin [9]:

Theorem 2.4 (Rudin)**.** *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. Suppose that e_1 is a boundary regular fixed point for f. Then $K\text{-}\lim_{z \to e_1} f(z) = e_1$. Moreover,*

($1'$) *$\langle df_z(e_1), e_1 \rangle$ and $\langle df_z(e_h), e_k \rangle$ are bounded in any Korányi region for $h, k = 2, \ldots, n$.*
($1''$) *$\langle df_z(e_j), e_1 \rangle / (1 - z_1)^{1/2}$ is bounded in any Korányi region for $j = 2, \ldots, n$.*

(1‴) $(1-z_1)^{1/2}\langle df_z(e_1), e_j\rangle$ *is bounded in any Korányi region for* $j = 2, \ldots, n$.
(2) $\angle_K \lim_{z\to e_1} \frac{1-\langle f(z), e_1\rangle}{1-z_1} = \alpha_f(e_1)$,
(3) $\angle_K \lim_{z\to e_1} \langle df_z(e_1), e_1\rangle = \alpha_f(e_1)$,
(4) $\angle_K \lim_{z\to e_1} \langle df_z(e_j), e_1\rangle = 0$ *for* $j = 2, \ldots, n$.
(5) $\angle_K \lim_{z\to e_1} \frac{\langle f(z), e_j\rangle}{(1-z_1)^{1/2}} = 0$ *for* $j = 2, \ldots, n$.
(6) $\angle_K \lim_{z\to e_1} (1-z_1)^{1/2}\langle df_z(e_1), e_j\rangle = 0$ *for* $j = 2, \ldots, n$.

2.1. **Automorphisms of $\mathbb{B}^n$.** Automorphisms of the unit ball are linear fractional maps of $\mathbb{C}^n$ which maps $\mathbb{B}^n$ onto $\mathbb{B}^n$ (see, *e.g.* [9, Section 2.2]). In particular they extend holomorphically past the boundary of the unit ball and they map the intersection of the unit ball with a given complex line onto the intersection of the unit ball with another complex line (for details see [5]). In what follows we need to consider two special families of automorphisms of the unit ball. Namely, we will use a particular type of *hyperbolic automorphisms* given by

$$\Phi_{t_0}(z) = \frac{(\cosh t_0\, z_1 + \sinh t_0, z_2, \ldots, z_n)}{\sinh t_0\, z_1 + \cosh t_0} \tag{2.3}$$

where $t_0 \in \mathbb{R} \setminus \{0\}$. A direct computation shows that $\Phi_{t_0}(e_1) = e_1$, that $\Phi_{t_0}(-e_1) = -e_1$ (and hence $\Phi_{t_0}(\mathbb{D} \times \{0\}) = \mathbb{D} \times \{0\}$) and $\alpha_{\Phi_{t_0}}(e_1) = e^{-2t_0}$. Moreover, $\det d(\Phi_{t_0})_0 = (\cosh t_0)^{-(n+1)}$. Note also that given $r \in (-1, 1)$, setting $t_0 = \frac{1}{2}\log\frac{1+r}{1-r}$, it follows that $\Phi_{t_0}(0) = re_1$.

Also, we will make use of the *parabolic* automorphisms of $\mathbb{B}^n$. Recall that an automorphism T of the unit ball fixing e_1 and with $\alpha_T(e_1) = 1$ is called a *parabolic automorphism* (fixing e_1).

Let $\mathbb{H}^n := \{(w_1, w'') \in \mathbb{C} \times \mathbb{C}^{n-1} : \mathsf{Re}\, w_1 > \|w''\|^2\}$ be the *Siegel domain*. The (generalized) *Cayley transform* $C : \mathbb{B}^n \to \mathbb{H}^n$ defined by $C(z_1, z'') = (1 + z_1, z'')/(1 - z_1)$, where as usual we set $z'' = (z_2, \ldots, z_n)$, is a biholomorphism. By [5, Proposition 4.3], if T is any parabolic automorphism of $\mathbb{B}^n$ fixing e_1 then

$$C \circ T \circ C^{-1}(w_1, w'') = (w_1 + 2\langle Uw'', a\rangle + c, Uw'' + a), \tag{2.4}$$

where U is a $(n-1) \times (n-1)$ unitary matrix, $a \in \mathbb{C}^{n-1}$ and $\mathsf{Re}\, c = \|a\|^2$.

Given $z_0 \in \mathbb{B}^n$, let $R(z_0) := \frac{|1-\langle z_0, e_1\rangle|^2}{1-\|z_0\|^2}$. Then, there exists a parabolic automorphism T_{z_0} such that $T_{z_0}(z_0) = \frac{1-R(z_0)}{1+R(z_0)}e_1$. Indeed, set $w_0 = C(z_0)$, $U = \mathsf{id}$, $a = -w_0''$, $\mathsf{Re}\, c = \|w_0''\|^2$ and $\mathsf{Im}\, c = -\mathsf{Im}\,(w_0)_1$ in the right-hand side of (2.4) and call $\tilde{T}_{z_0}(w)$ such a map. Then

$$\begin{aligned} T_{z_0}(z_0) &= C^{-1}(\tilde{T}_{z_0}(C(z_0)) = C^{-1}(\mathsf{Re}\,(w_0)_1 - \|w_0''\|^2, 0, \ldots, 0) \\ &= C^{-1}\left(\frac{1-\|z_0\|^2}{|1-\langle z_0, e_1\rangle|^2}, 0, \ldots, 0\right) = \frac{1-R(z_0)}{1+R(z_0)}e_1. \end{aligned}$$

A direct computation (or see [5, eq. (4.2)]) shows that for any parabolic automorphism T of the unit ball $\det dT_{e_1} = 1$.

2.2. **Range close to boundary regular fixed points.** Define the *Stolz angle*

$$K(M,s) := \{\zeta \in \mathbb{D} : \frac{|1-\zeta|}{1-|\zeta|} < M, |1-\zeta| < s\}.$$

Notice that $K(M,s)$ is the intersection of a Korányi region/a cone of vertex e_1 and amplitude M with the slice $\mathbb{C}e_1$ and the Euclidean ball $\mathbb{B}(e_1,s)$. In particular, if $g : \mathbb{B}^n \to \mathbb{C}^m$ is a holomorphic map such that $\angle\lim_{z\to e_1} g(z) = L$, then $\lim_{K(M,s)\ni\zeta\to 1} g(\zeta e_1) = L$.

As a matter of notation, let $K(M,s)e_1 := \{z \in \mathbb{B}^n : z = \zeta e_1, \zeta \in K(M,s)\}$

Proposition 2.5. *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. Assume that e_1 is a boundary regular fixed point of f. Then for every $\epsilon > 0$, $M > 1$, $t \in (0,1)$ there exists $s = s(\epsilon, M, t) > 0$ such that $k_{\mathbb{B}^n}(z, f(K(M,t)e_1))) < \epsilon$ for all $z \in K(M,s)e_1$.*

Proof. First of all, we assume that $\alpha_f(e_1) = 1$. For a given $s \in (0,1)$ consider the complex curve $K(M,s) \ni \zeta \mapsto f(\zeta e_1) \in \mathbb{B}^n$. By Theorem 2.4, it follows that such a curve is continuous up to the closure of $K(M,s)$.

We claim that, for each $M > 1$ and each $\epsilon > 0$ there exists $s_0 = s_0(M,\epsilon) \in (0,1)$ such that for all fixed $s \in (0,s_0)$ it follows

$$k_{\mathbb{B}^n}(f(\zeta e_1), \pi(f(\zeta e_1))) < \frac{\epsilon}{2} \quad \forall \zeta \in K(M,s). \tag{2.5}$$

Thanks to (2.1), claim (2.5) is equivalent to the claim that for each $M > 1$ and each $\delta > 0$ there exists $s_0 = s_0(M,\delta) \in (0,1)$ such that for all fixed $s \in (0,s_0)$ it holds

$$\frac{\|f(\zeta e_1) - \pi(f(\zeta e_1))\|^2}{1 - \|\pi(f(\zeta e_1))\|^2} < \delta \quad \forall \zeta \in K(M,s). \tag{2.6}$$

In order to prove this, we write

$$\begin{aligned}\frac{\|f(\zeta e_1) - \pi(f(\zeta e_1))\|^2}{1 - \|\pi(f(\zeta e_1))\|^2} &= \frac{\|f(\zeta e_1) - \pi(f(\zeta e_1))\|^2}{|(1-\zeta)^{1/2}|^2} \cdot \frac{|1-\zeta|}{|1-f_1(\zeta e_1)|} \cdot \frac{|1-f_1(\zeta e_1)|}{1-|f_1(\zeta e_1)|^2} \\ &=: h_1(\zeta)\cdot h_2(\zeta)\cdot h_3(\zeta).\end{aligned}$$

By Theorem 2.4.(2) and (5), the functions $\overline{K(M,s)} \ni \zeta \mapsto h_1(\zeta)$ and $K(M,s) \ni \zeta \mapsto h_2(\zeta)$ are (uniformly) continuous on the compact set $\overline{K(M,s)}$. Moreover, $\lim_{K(M,s)\ni\zeta\to 1} h_1(\zeta) = 0$, while $\lim_{K(M,s)\ni\zeta\to 1} h_2(\zeta) = 1$. As for the function h_3, we notice that the function $g(\zeta) := f_1(\zeta e_1)$ is a holomorphic self-map of the unit disc, having 1 as a boundary regular fixed point because of Theorem 2.4.(2) and since

$$\alpha_g(1) \leq \liminf_{r\to 1} \frac{1-|g(r)|}{1-r} \leq \liminf_{r\to 1} \frac{|1-f_1(re_1)|}{1-r} = 1.$$

Therefore for any $\zeta \in \mathbb{D}$ such that $\frac{|1-\zeta|}{1-|\zeta|} < M$,

$$\frac{|1-g(\zeta)|}{1-|g(\zeta)|} = \frac{|1-g(\zeta)|}{|1-\zeta|}\frac{|1-\zeta|}{1-|\zeta|}\frac{1-|\zeta|}{1-|g(\zeta)|}$$

is bounded from above by a constant depending only on M because of the classical Julia-Wolff-Carathéodory (that is, Theorem 2.4 for $n = 1$). Hence, $h_3(\zeta)$ is bounded from above. Thus, (2.6) follows.

Next, we claim that for each $M > 1$ and each $\epsilon > 0$ there exists $s_1 = s_1(M, \epsilon) \in (0, 1)$ such that for all fixed $s \in (0, s_1)$ it follows

$$k_{\mathbb{B}^n}(\zeta e_1, \pi(f(\zeta e_1))) < \frac{\epsilon}{2} \quad \forall \zeta \in K(M, s). \tag{2.7}$$

Let $h(\zeta) := f_1(\zeta e_1)$. Thus $h : \mathbb{D} \to \mathbb{D}$ is a holomorphic function and, as before, 1 is a boundary regular fixed point of h. Moreover, from Theorem 2.4.(2) and since we assumed $\alpha = 1$, it follows that $\alpha_h(1) = 1$. Now,

$$k_{\mathbb{B}^n}(\zeta e_1, \pi(f(\zeta e_1))) = k_{\mathbb{D}}(\zeta, h(\zeta)) = \frac{1}{2} \log \frac{1 + \left|\frac{\zeta - h(\zeta)}{1 - \overline{\zeta} h(\zeta)}\right|}{1 - \left|\frac{\zeta - h(\zeta)}{1 - \overline{\zeta} h(\zeta)}\right|},$$

hence (2.7) follows as soon as we prove that for each $M > 1$ and each $\delta > 0$ there exists $s_1 = s_1(M, \delta) \in (0, 1)$ such that for all fixed $s \in (0, s_1)$ it holds

$$\left|\frac{\zeta - h(\zeta)}{1 - \overline{\zeta} h(\zeta)}\right| < \delta \quad \forall \zeta \in K(M, s). \tag{2.8}$$

Let $d(\zeta) := (1 - h(\zeta))/(1 - \zeta)$ and $a(\zeta) = (1 - \overline{\zeta})/(1 - \zeta)$. Thus

$$\left|\frac{\zeta - h(\zeta)}{1 - \overline{\zeta} h(\zeta)}\right| = \left|\frac{1 - d(\zeta)}{a(\zeta) + \overline{\zeta} d(\zeta)}\right|. \tag{2.9}$$

By Theorem 2.4 for $n = 1$ it follows that $d(\zeta)$ is (uniformly) continuous on the compact set $\overline{K(M, s)}$ and $d(\zeta) \to 1$ as $\overline{K(M, s)} \ni \zeta \to 1$. Moreover, if $s < 1/2$ then $\mathsf{Re}\, \zeta > 1/2$ and then $\mathsf{Re}\, a(\zeta) > -1/4$. Hence, if s is sufficiently small, it follows that the real part of the denominator of the right hand side of (2.9) is bounded from below by a constant $c > 0$ for all $\zeta \in K(M, s)$. As a result, if we choose s sufficiently small, (2.8) holds.

Now, fix $\epsilon > 0$, $M > 1$ and $t \in (0, 1)$. Choose s_0, s_1 such that (2.5), (2.7) hold, and let $s := \min\{t, s_0, s_1\}$. Let $z \in K(M, s)e_1$, hence

$$k_{\mathbb{B}^n}(f(K(M, s)e_1), z) \le k_{\mathbb{B}^n}(f(z), z) \le k_{\mathbb{B}^n}(z, \pi(f(z))) + k_{\mathbb{B}^n}(\pi(f(z)), f(z)) < \epsilon.$$

To end up the proof, we need to consider the case $\alpha := \alpha_f(e_1) \neq 1$. Let Φ be an *hyperbolic* automorphism of $\mathbb{B}^n$ of type (2.3) with $\alpha_\Phi(e_1) = \alpha^{-1}$. In particular, since the first component is a Möbius transformation of $\mathbb{C}$, it follows that for all $s \in (0, 1)$ there exists $s'(s) \in (0, 1)$ such that $\Phi(K(M, s'(s))e_1) = K(M, s)e_1$. Using Theorem 2.4 it is easy to see that the holomorphic self-map $g := f \circ \Phi$ of $\mathbb{B}^n$ has the property that e_1 is a boundary regular fixed point of g and $\alpha_g(e_1) = 1$. Let $\epsilon > 0, t \in (0, 1)$ and $M > 1$. Let $t'(t) \in (0, 1)$ be such that $\Phi(K(M, t'(t))e_1) = K(M, t)e_1$. We apply the result already proven to g, finding $s > 0$ (in fact, for what we have proven, $s = t'(t)$) such that $k_{\mathbb{B}^n}(z, g(K(M, t'(t))e_1)) < \epsilon$ for all $z \in K(M, s)$. By construction $g(K(M, t'(t))e_1) =$

$f(\Phi(K(M,t'(t))e_1) = f(K(M,t)e_1)$ and therefore $k_{\mathbb{B}^n}(z, f(K(M,t)e_1) < \epsilon$ for all $z \in K(M,s)e_1$, as wanted. □

Corollary 2.6. *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. Assume that e_1 is a boundary regular fixed point of f. Then for every $\epsilon > 0$ and $t \in (0,1)$ and for every $A \subset \mathbb{B}^n$ admissible at e_1 there exists $\delta > 0$ such that $k_{\mathbb{B}^n}(z, f(K(M,t)e_1)) < \epsilon$ for all $z \in A \cap \mathbb{B}(e_1, \delta)$.*

Proof. Fix $\epsilon > 0$ and $t \in (0,1)$. Let $\delta_1 > 0$ be such that $\pi(A \cap \mathbb{B}(e_1, \delta_1)) \subset K(M, s_1)e_1$ for some $s_1 \in (0,1)$ and $k_{\mathbb{B}^n}(z, \pi(z)) \leq \epsilon/2$ for all $z \in A \cap \mathbb{B}(e_1, \delta_1)$. Let $s_0 \in (0,1)$ be such that $k_{\mathbb{B}^n}(z, f(K(M,t))) \leq \epsilon/2$ for all $z \in K(M, s_0)e_1$. Let $\delta := \min\{t, s_0, \delta_1\}$. Let $z \in A \cap \mathbb{B}(e_1, \delta) \subset A \cap \mathbb{B}(e_1, \delta_1)$. Hence $\pi(z) \in K(M, s_0)$ and

$$k_{\mathbb{B}^n}(z, f(K(M,t)e_1)) \leq k_{\mathbb{B}^n}(z, \pi(z)) + k_{\mathbb{B}^n}(\pi(z), f(K(M,t)e_1)) < \epsilon,$$

and the proof is completed. □

Example 2.7. In general, Corollary 2.6 is the best one can say, namely, the image of a holomorphic map close to a boundary regular fixed point can be very thin: just consider the simple example $f : \mathbb{B}^2 \to \mathbb{B}^2$ given by $f(z_1, z_2) = (z_1, 0)$. Even assuming univalency the situation does not improve: the map $f(z_1, z_2) = (z_1, \frac{1}{4}z_2(1 - z_1)^2)$ is a univalent self-map of $\mathbb{B}^2$ having a boundary regular fixed point at e_1 but its image does not contain eventually any cone with vertex at e_1.

The problem with the previous examples is that the Jacobian of the maps at e_1 is zero. In the next section we show that if this is not the case, then the range of the map close to the boundary point is "fat".

2.3. Range close to boundary super-regular fixed points.

Definition 2.8. Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. The point e_1 is a *boundary super-regular fixed point* of f if it is a boundary regular fixed point of f and moreover for each $M > 1$ there exists $c = c(M) > 0$ such that for any sequence $\{\zeta_k\} \subset K(M, 2)$ which tends to 1 it holds $\liminf_{k\to\infty} |\det(df_{\zeta_k e_1})| \geq c$.

Remark 2.9. Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. Assume e_1 is a boundary regular fixed point of f. As shown in [9, Example p.183], the radial limit of $\det(df_z)$ at e_1 may not exist. In fact, by Theorem 2.4 it follows that $\det df_z$ is bounded in every Korányi region and Čirca's theorem [9, Theorem 8.4.8] implies that if $\lim_{r\to 1} \det(df_{re_1})$ exists, then $\angle_K \lim_{z\to e_1} \det df_z$ exists. If this is the case, e_1 is a boundary super-regular fixed point if and only if $\angle_K \lim_{z\to e_1} \det df_z \neq 0$.

Lemma 2.10. *Let $c > 0$. Let $\mathcal{F}_c := \{f : \mathbb{B}^n \to \mathbb{B}^n \text{ holomorphic} : f(0) = 0, |\det df_0| \geq c\}$. Let $t' \in (0,1)$. Then there exists $r' = r'(c, t') > 0$ such that $\mathbb{B}(0, r') \subset f(\mathbb{B}(0, t'))$ for all $f \in \mathcal{F}_c$.*

Proof. Assume this is not the case. Then for all $n \in \mathbb{N}$ such that $1/n < t'$ there exists $z_n \in \mathbb{B}(0,t')$ with $|z_n| < 1/n$ and $f_n \in \mathcal{F}_c$ such that $z_n \not\in f(\mathbb{B}(0,t'))$. Since $\|f_n(z)\| < 1$ for all $z \in \mathbb{B}^n$, and $f_n(0) = 0$, $\{f_n\}$ is a normal family and we can assume it is uniformly convergent on compacta to a holomorphic map $f : \mathbb{B}^n \to \mathbb{B}^n$ such that $f(0) = 0$. Since $d(f_n)_0 \to df_0$, the map $f \in \mathcal{F}_c$. Therefore there exists $\delta > 0$ such that $\mathbb{B}(0,\delta) \subset f(\mathbb{B}(0,t'))$. Hence, eventually $\mathbb{B}(0,\delta/2) \subset f_n(\mathbb{B}(0,t'))$, which contradicts the choice of $\{z_n\}$. $\square$

Theorem 2.11. *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. Assume that e_1 is a boundary super-regular fixed point of f. Then for every $\eta > 0$ and $M > 1$ there exist $s \in (0,1)$ and $r > 0$ such that*

$$\bigcup_{\zeta \in K(M,s)} B_k(f(\zeta e_1), r)) \subset f(\mathbb{B}^n \cap \mathbb{B}(e_1,\eta)), \tag{2.10}$$

where $B_k(x,R)$ denotes the Kobayashi ball of center $x \in \mathbb{B}^n$ and radius $R > 0$.

Proof. For $z \in \mathbb{B}^n$, let us set

$$R(z) = \frac{|1-z_1|^2}{1-\|z\|^2}, \quad r(z) = \frac{1-R(z)}{1+R(z)}.$$

Let $\zeta \in K(M,s)$, for some $s \in (0,1)$ to be chosen later. Let S be a parabolic automorphism of $\mathbb{B}^n$ fixing e_1 such that $S(\zeta e_1) = r(\zeta e_1)e_1$. Next, let Φ_{t_0} be a hyperbolic automorphism of the form (2.3) with $t_0 = \frac{1}{2}\log\frac{1+r(\zeta e_1)}{1-r(\zeta e_1)} = -\log R(\zeta e_1)^{1/2}$. Hence $\Phi_{t_0}(0) = r(\zeta e_1)e_1$.

let T be a parabolic automorphism of $\mathbb{B}^n$ fixing e_1 such that $T(f(\zeta e_1)) = r(f(\zeta e_1))e_1$, and let Φ_{t_1} be a hyperbolic automorphism of the form (2.3) with $t_1 = \frac{1}{2}\log\frac{1+r(f(\zeta e_1))}{1-r(f(\zeta e_1))} = -\log R(f(\zeta e_1))^{1/2}$. Hence $\Phi_{t_1}(0) = r(f(\zeta e_1))e_1$.

Now, let $g^\zeta := \Phi_{t_1}^{-1} \circ T \circ f \circ S^{-1} \circ \Phi_{t_0}$. By construction, $g^\zeta : \mathbb{B}^n \to \mathbb{B}^n$ is holomorphic and $g^\zeta(0) = 0$. Moreover,

$$\det dg_0^\zeta = \det(\Phi_{t_1}^{-1})_{r(f(\zeta e_1))e_1} \cdot \det dT_{f(\zeta e_1)} \cdot \det df_{\zeta e_1} \cdot \det dS^{-1}_{r(\zeta e_1)e_1} \cdot \det d(\Phi_{t_0})_0. \tag{2.11}$$

Since f is (uniformly) continuous on $\overline{K(M,s)}$ for any $s \in (0,1)$ and $\lim_{K(M,s)\ni\zeta\to 1} f(\zeta e_1) = e_1$, and since $\det dT_{e_1} = \det dS_{e_1} = 1$, it follows that there exists $s_0 \in (0,1)$ such that for all $s \in (0,s_0)$ and for each $\zeta \in K(M,s)$ it holds

$$|\det dS_{\zeta e_1}| \le 2, \quad |\det dT_{f(\zeta e_1)}| \ge \frac{1}{2}. \tag{2.12}$$

Therefore, since $\det dS^{-1}_{r(\zeta e_1)e_1} = \det dS^{-1}_{S(\zeta e_1)} = (\det dS_{\zeta e_1})^{-1}$, it follows that

$$|\det dS^{-1}_{r(\zeta e_1)e_1}| \ge \frac{1}{2}. \tag{2.13}$$

Also,

$$\begin{aligned}\det(\Phi_{t_1}^{-1})_{r(f(\zeta e_1))e_1} &= \det d(\Phi_{t_1}^{-1})_{\Phi_{t_1}(0)} = (\det(d\Phi_{t_1})_0)^{-1} = (\cosh t_1)^{n+1},\\ \det d(\Phi_{t_0})_0 &= (\cosh t_0)^{-(n+1)}.\end{aligned} \tag{2.14}$$

Finally, taking into account that e_1 is a boundary super-regular fixed point of f, there exists $s_1 \in (0, s_0)$, and $c > 0$ such that $|df_{\zeta e_1}| \geq c$ for all $\zeta \in K(M, s) \subset K(M, 2)$ for all $s \in (0, s_1)$. From (2.11), (2.12), (2.13), (2.14) we obtain

$$|\det dg_0^\zeta| \geq \frac{c}{4}\left(\frac{\cosh t_1}{\cosh t_0}\right)^{n+1}.$$

A direct computation shows that, for $\zeta \in K(M, s)$,

$$\begin{aligned}\frac{\cosh t_1}{\cosh t_0} &= \frac{1 + R(f(\zeta e_1))}{1 + R(\zeta e_1)} \cdot \left(\frac{R(\zeta e_1)}{R(f(\zeta e_1))}\right)^{1/2}\\ &= \frac{\frac{1-\|f(\zeta e_1\|^2}{1-|\zeta|^2} + \frac{|1-f_1(\zeta e_1)|^2}{1-|\zeta|^2}}{1 + \frac{|1-\zeta|^2}{1-|\zeta|^2}} \frac{1-|\zeta|^2}{1-\|f(\zeta e_1)\|^2}\left(\frac{|1-\zeta|^2}{|1-f_1(\zeta e_1)|^2}\frac{1-\|f(\zeta e_1)\|^2}{1-|\zeta|^2}\right)^{1/2}\\ &\geq \frac{\frac{1-\|f(\zeta e_1\|^2}{1-|\zeta|^2}}{1+M|1-\zeta|}\frac{1-|\zeta|^2}{1-\|f(\zeta e_1)\|^2}\frac{|1-\zeta|}{|1-f_1(\zeta e_1)|}\left(\frac{1-\|f(\zeta e_1)\|^2}{1-|\zeta|^2}\right)^{1/2}\\ &\geq \frac{1}{1+Ms}\frac{|1-\zeta|}{|1-f_1(\zeta e_1)|}\left(\frac{1-\|f(\zeta e_1)\|^2}{1-|\zeta|^2}\right)^{1/2}\end{aligned} \tag{2.15}$$

Let $\alpha := \alpha_f(e_1)$. Since $\alpha = \liminf_{z\to e_1}(1-\|f(z)\|)/(1-\|z\|)$, there exists $s_2 \in (0, s_1)$ such that for all $s \in (0, s_1)$, it holds $\frac{1-\|f(\zeta e_1)\|^2}{1-|\zeta|^2} \geq \alpha/4$ for all $\zeta \in K(M, s)$.

Also, since e_1 is a boundary regular fixed point for f by Theorem 2.4.(2) it follows that there exists $s_3 \in (0, s_2)$ such that for all $s \in (0, s_3)$, it holds $\frac{|1-f_1(\zeta e_1)|}{|1-\zeta|} \leq 2\alpha$ for all $\zeta \in K(M, s)$. Using the previous estimates in (2.15), we conclude that there exists $c' > 0$ such that $\left(\frac{\cosh t_1}{\cosh t_0}\right)^{n+1} \geq c'$ and hence $|\det dg_0^\zeta| \geq c'' := cc'/4$.

Therefore, for all $\zeta \in K(M, s_3)$, the map $g^\zeta \in \mathcal{F}_{c''}$, where $\mathcal{F}_{c''}$ is defined in Lemma 2.10. Thus, by Lemma 2.10, for every $t' \in (0, 1)$ there exists $r' \in (0, 1)$ such that $\mathbb{B}(0, r') \subset g^\zeta(\mathbb{B}(0, t'))$ for all $\zeta \in K(M, s_3)$.

Now, for $a' > 0$ set $a = \frac{1}{2}\log\frac{1+a'}{1-a'}$. Since automorphisms of $\mathbb{B}^n$ are isometries for $k_{\mathbb{B}^n}$ and $\mathbb{B}(0, a') = B_k(0, a)$, by the very definition of g^ζ it follows that for all $\zeta \in K(M, s_3)$

$$B_k(f(\zeta e_1), r) = T^{-1}(\Phi_{t_1}(\mathbb{B}(0, r')) \subset f(S^{-1}(\Phi_{t_0}(\mathbb{B}(0, t'))) = f(B_k(\zeta e_1, t)).$$

Finally, let $\eta > 0$ be as in claim (2.10) and choose $s \in (0, s_3)$ such that for each $\zeta \in K(M, s)$, it holds $B_k(\zeta e_1, t) \subset \mathbb{B}^n \cap \mathbb{B}(e_1, \eta)$. This can be done because the Euclidean

diameter of a Kobayashi ball of fixed radius tends to zero when the center tends to e_1. Hence for any $\zeta \in K(M,s)$,

$$B_k(f(\zeta e_1), r) \subset f(B_k(\zeta e_1, t)) \subset f(\mathbb{B}^n \cap \mathbb{B}(e_1, \eta)),$$

and (2.10) is proved. $\square$

As a consequence, we have that any subset $A \subset \mathbb{B}^n$ which is non-tangentially asymptotic at e_1 is eventually contained in the range close to a boundary super-regular fixed point:

Theorem 2.12. *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. Assume that e_1 is a boundary super-regular fixed point of f. Then for every $\eta > 0$ and for every $A \subset \mathbb{B}^n$ admissible at e_1 there exists $\delta > 0$ such that $A \cap \mathbb{B}(e_1, \delta) \subset f(\mathbb{B}(e_1, \eta) \cap \mathbb{B}^n)$.*

Proof. Let $\eta > 0$. Let $s, r > 0$ be given by Theorem 2.11. Let $0 < \epsilon < r$. By Corollary 2.6 there exists $\delta > 0$ such that $k_{\mathbb{B}^n}(z, f(K(M,s)e_1)) < \epsilon$ for all $z \in A \cap \mathbb{B}(e_1, \delta)$. Therefore, any $z \in A \cap \mathbb{B}(e_1, \delta)$ is contained in a Kobayashi ball centered at some point of $f(K(M,s)e_1)$ and with radius at most $\epsilon < r$, hence by Theorem 2.11, it is contained in $f(\mathbb{B}^n \cap \mathbb{B}(e_1, \eta))$, as stated. $\square$

3. Strongly pseudoconvex domains

Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with smooth boundary. Let $p \in \partial D$. We denote by ν_p^D the outer unit normal vector to ∂D at p. For $z, w \in \mathbb{C}^n$, let $d(z, w) = \|z - w\|$. A *cone* $C(p, M)$ with vertex p and amplitude $M > 1$ is defined as

$$C(p, M) := \{z \in D : d(p, z) < M d(z, \partial D)\}.$$

We say that a sequence $\{z_k\} \subset D$ tends to p *normally non-tangentially* if $z_k \to p$ and for all $k \in \mathbb{N}$ there exists $\zeta_k \in \mathbb{C}$ such that $z_k = \zeta_k \nu_p^D$ and there exists $M > 1$ such that $\{z_k\} \subset C(p, M)$.

Definition 3.1. Let $D, D' \subset \mathbb{C}^n$ be two bounded strongly pseudoconvex domains with smooth boundary. Let $f : D \to D'$ be holomorphic. A point $p \in \partial D$ is said to be a *regular contact point* for f if

$$\liminf_{D \ni z \to p} \frac{d(f(z), \partial D')}{d(z, \partial D)} < +\infty.$$

The point p is said to be a *super-regular contact point* for f if it is a regular contact point for f and moreover for every $M > 1$ there exists $c = c(M) > 0$ such that

$$\liminf_{k \to \infty} |\det df_{z_k}| \geq c,$$

for every sequence $\{z_k\} \subset C(M, p)$ which converges normally non-tangentially to p.

In [3, Theorem 0.2] M. Abate proved that if $f : D \to D'$ is a holomorphic map between two bounded strongly pseudoconvex domains and $p \in \partial D$ is a regular contact point, then an analogous of Rudin's theorem 2.4 holds. In the proof of our main Theorem 3.6 we will make use of part of Abate's theorem. For the reader convenience we state here what we need from [3, Theorem 0.2]:

Theorem 3.2 (Abate). *Let $D, D' \subset \mathbb{C}^n$ be two bounded strongly pseudoconvex domains with smooth boundary. Let $f : D \to D'$ be holomorphic. Assume $p \in \partial D$ is a regular contact point for f. Then there exists $q \in \partial D'$ such that*

(1) $\lim_{\in(0,1)\ni r\to 0} f(p - r\nu_p^D) = q$,
(2) $\lim_{\in(0,1)\ni r\to 0} \langle df_{p-r\nu_p^D}(\nu_p^D), \nu_q^{D'}\rangle$ *exists finitely.*

Also, we will make use of the following recent result by the second named author with E. F. Wold and K. Diederich [8]:

Theorem 3.3. *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with smooth boundary and let $q \in \partial D$. Then there exists a biholomorphism Φ from an open neighborhood of $\overline{D}$ such that $\Phi(D) \subset \mathbb{B}^n$ and $\Phi(q) = e_1$.*

Definition 3.4. Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with smooth boundary. A subset $A \subset D$ such that $p \in \overline{A}$ is *admissible* at $p \in \partial D$ if for every $\epsilon > 0$ there exist $\delta > 0$ and $M > 1$ such that

(1) $\langle z, \nu_p^D\rangle\nu_p^D \in C(p, M)$ for all $z \in A \cap \mathbb{B}(p, \delta)$,
(2) $k_D(z, \pi(z)) < \epsilon$ for all $z \in A \cap \mathbb{B}(p, \delta)$,

where k_D denotes the Kobayashi distance in D.

Lemma 3.5. *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with smooth boundary. Any cone $C(p, M) \subset D$ with vertex $p \in \partial D$ and amplitude $M > 1$ is admissible at p.*

Proof. It is enough to note that if $B \subset D$ is a Euclidean ball tangent to p, then every cone $C(p, M)$ is eventually contained in B and $C(p, M) \cap B$ is contained in a cone in B with vertex p. Hence by Lemma 2.3 it is admissible in B. By the monotonicity property of the Kobayashi distance, $k_D \le k_B$ and hence the result follows. □

Our main result is the following:

Theorem 3.6. *Let $D, D' \subset \mathbb{C}^n$ be two bounded strongly pseudoconvex domains with smooth boundary. Let $f : D \to D'$ be holomorphic. Assume $p \in \partial D$ is a super-regular contact point for f. Then there exists a point $q \in \partial D'$ such that for every $\eta > 0$ and for every $A \subset D'$ admissible at q there exists $\delta > 0$ such that $A \cap \mathbb{B}(q, \delta) \subset f(\mathbb{B}(p, \eta) \cap D)$.*

Proof. Up to rotations and dilations, we can assume that $p = e_1$ and $\mathbb{B}^n \subset D$. Moreover, by Theorem 3.3, we can assume that $q = e_1$ and $D' \subset \mathbb{B}^n$. In particular, with these choices, $\nu_p^D = \nu_q^{D'} = e_1$. Hence, the map $g := f|_{\mathbb{B}^n} : \mathbb{B}^n \subset D \to D' \subset \mathbb{B}^n$ is a holomorphic

self-map of $\mathbb{B}^n$ and by Theorem 3.2.(1), $\lim_{\in(0,1)\ni r\to 1} g(re_1) = e_1$. We claim that e_1 is a boundary super-regular fixed point for g. Since e_1 is a super-regular contact point for $f : D \to D'$, in order to show that e_1 is a boundary super-regular fixed point for g, we only need to show that

$$\liminf_{z\to e_1} \frac{1-\|g(z)\|}{1-\|z\|} < +\infty. \tag{3.1}$$

By Theorem 3.2.(2), we have

$$\lim_{(0,1)\ni r\to 1} \langle dg_{re_1}(e_1), e_1\rangle = \lim_{(0,1)\ni r\to 0} \langle df_{p-r\nu_p^D}(\nu_p^D), \nu_q^{D'}\rangle = L,$$

for some $L \in \mathbb{C}$. Let $h(\zeta) := \langle g(\zeta e_1), e_1\rangle$. Hence h is a holomorphic self-map of $\mathbb{D}$ and $\lim_{r\to 1} h(r) = 1$, $\lim_{r\to 1} h'(r) = \lim_{(0,1)\ni r\to 1}\langle dg_{re_1}(e_1), e_1\rangle = L$. By the mean value theorem applied to the real and imaginary part of $(0,1) \ni r \mapsto h(r)$, it follows that

$$\lim_{r\to 1} \frac{1-h(r)}{1-r} = L.$$

Therefore

$$\begin{aligned}\liminf_{z\to e_1} \frac{1-\|g(z)\|}{1-\|z\|} &\leq \liminf_{(0,1)\ni r\to 1} \frac{1-\|g(re_1)\|}{1-r} \leq \liminf_{(0,1)\ni r\to 1} \frac{1-|\langle g(re_1), e_1\rangle|}{1-r} \\ &\leq \liminf_{(0,1)\ni r\to 1} \frac{|1-h(r)|}{1-r} = |L| < +\infty,\end{aligned}$$

as needed.

Now, note that A is admissible at e_1 not only as a subset of D but also as a subset of $\mathbb{B}^n$. Indeed, its projection into $\mathbb{C}e_1$ is eventually contained in a cone in D and hence in $\mathbb{B}^n$, also, the Kobayashi distance is monotonic and then $k_{\mathbb{B}^n}(z, \pi(z)) \leq k_D(z, \pi(z)) < \epsilon$ for all $z \in A$ close to e_1. Thus, we can apply Theorem 2.12 to g and, since $g(\mathbb{B}^n \cap \mathbb{B}(e_1, \eta)) \subset f(D \cap \mathbb{B}(e_1, \eta))$ for all $\eta > 0$ we get also the result for f. □

Remark 3.7. On the one hand, it would be interesting to give a direct proof of Theorem 3.6 without using the embedding Theorem 3.3, but using instead the full Abate's version of Rudin's theorem for strongly pseudoconvex domains and suitably adapting our proof of Theorem 2.12. However, aside using Rudin's theorem, our argument in the proof of Theorem 2.12 is strongly based on the existence of a family of good automorphisms of $\mathbb{B}^n$, and it is not clear how to bypass such an argument for strongly pseudoconvex domains.

On the other hand, it would be interesting to prove (3.1) without using Abate's Theorem 3.2. If this were possible, the previous method would allow to prove Abate's version of Rudin's theorem for strongly pseudoconvex domains directly by means of Rudin's theorem.

4. Images of regular contact points by univalent mappings

Theorem 4.1. *Let $D, D' \subset \mathbb{C}^n$ be bounded strongly pseudoconvex domains with smooth boundary. Let $f : D \to D'$ be a univalent map and assume $x \in \partial D$ is a super-regular contact point for f, and denote by $f(x) \in \partial D'$ the non-tangential limit of f at x. If $y \in \partial D$ is a regular fixed point for f then $f(x) \neq f(y)$.*

Proof. Assume by contradiction that $f(x) = f(y)$. Up to rotations and dilations, we can assume that $y = e_1$ and $\mathbb{B}^n \subset D$. Moreover, by Theorem 3.3, we can assume that $f(x) = e_1$ and $D' \subset \mathbb{B}^n$. Let $g := f|_{\mathbb{B}^n} : \mathbb{B}^n \subset D \to D' \subset \mathbb{B}^n$, seen as a holomorphic self-map of the unit ball. Arguing as in the proof of Theorem 3.6, we see that e_1 is a boundary regular fixed point for g. Hence, the curve $\Gamma : (0,1) \ni r \mapsto g(re_1)$ is admissible at e_1 by [4, Lemma 2.2]. In fact, g maps Korányi regions into Korányi regions because e_1 is a boundary regular fixed point (see, *e.g.*, [9, Theorem 8.5.4]), hence $(0,1) \ni r \mapsto \pi(g(re_1))$ is non-tangential and it is asymptotic by (2.5).

Let $B \subset D$ be an Euclidean ball tangent to ∂D at p. Consider $h = f|_B : B \subset D \to D' \subset \mathbb{B}^n$ as a holomorphic map from B to $\mathbb{B}^n$. By Theorem 3.6 it follows that for every $\eta > 0$ there exists $\delta > 0$ such that $\Gamma \cap \mathbb{B}(e_1, \delta) \subset h(B \cap \mathbb{B}(p, \eta)) \subset f(D \cap \mathbb{B}(p, \eta))$. For η small, this contradicts the univalency of f. $\square$

In dimension one, regular contact points and super-regular contact points are one and the same, and Theorem 4.1 recovers [7, Lemma 8.2]. In higher dimensions however, a univalent mapping can send two different regular contact points (but not super-regular) to the same point, as the following example shows:

Example 4.2. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic mapping with the following properties:

(1) $\varphi(\mathbb{D}) \subset \{\zeta \in \mathbb{D} : |\zeta - 1| < 1/2\}$,
(2) φ extends smoothly up to the boundary and $\varphi(-1) = \varphi(1) = 1$,
(3) there exist $0 < r_{-1}, r_1 < 1/2$ such that the two open discs $A = D(-1, r_{-1}), B = D(1, r_1)$ are disjoint and φ restricted to $\mathbb{D} \setminus A$ and restricted to $\mathbb{D} \setminus B$ is univalent.

Such a map can be constructed by taking the Riemann mapping from the unit disc to an open simply connected Riemann surface with smooth boundary in $\mathbb{C}^4$ which sits on a helicoid a project the image back to $\mathbb{C}$.

Now, let $C : \mathbb{D} \to \mathbb{H} := \{\zeta \in \mathbb{C} : \mathsf{Re}\,\zeta > 0\}$ be the Cayley transform $C(\zeta) = (1+\zeta)/(1-\zeta)$. Let $\phi := C \circ \varphi \circ C^{-1}$. Set $R_0 := r_{-1}/(2-r_{-1})$, $R_\infty = (2-r_1)/r_1$ and let $D(x, R)$ denote the Euclidean disc of center $x \in \mathbb{C}$ and radius $R > 0$. Then $U = C(A) = \mathbb{H} \cap D(0, R_0)$ and $V = C(B) = \mathbb{H} \setminus \overline{D(0, R_\infty)}$. The set $\phi(\mathbb{H})$ is contained in $\{\zeta \in \mathbb{C} : \mathsf{Re}\,\zeta > 3\}$ and ϕ is univalent in $\mathbb{H} \setminus U$ and in $\mathbb{H} \setminus V$. Moreover, $\lim_{\mathbb{H} \ni \zeta \to 0} |\phi(\zeta)| = \lim_{\mathbb{H} \ni \zeta \to \infty} |\phi(\zeta)| = \infty$. Let

$$\alpha := \lim_{(0,\infty) \ni r \to \infty} \frac{\phi(r)}{r}.$$

Such a limit exists, $\alpha > 0$ and moreover $\mathsf{Re}\,\phi(\zeta) > \alpha \mathsf{Re}\,\zeta$ for all $\zeta \in \mathbb{H}$ by the Julia-Wolff-Carathéodory theorem (that is Theorem 2.4 for $n = 1$ in its right-half plane formulation, see, *e.g.* [1, Corollary 1.2.12]).

Let $\mathbb{H}^2 := \{(z, w) \in \mathbb{C}^2 : \mathsf{Re}\,z > \|w\|^2\}$. For $\delta, \epsilon > 0$ we define a map $\Phi : \mathbb{H}^2 \to \mathbb{C}^2$ as follows

$$\Phi(z, w) = \left(\phi(z), \frac{\sqrt{\alpha}\delta w}{1 + \phi(z)} + \frac{\epsilon z}{1 + z}\right).$$

The map Φ is clearly holomorphic, and we claim that, for sufficiently small $\delta, \epsilon > 0$ it maps $\mathbb{H}^2$ into $\mathbb{H}^2$ and it is univalent. Assume this is the case, let $C(z, w) := (1 + z, w)/(1 - z)$ be the Cayley transform from $\mathbb{B}^2$ onto $\mathbb{H}^2$. Let $f := C^{-1} \circ \Phi \circ C$. Then $f : \mathbb{B}^2 \to \mathbb{B}^2$ is a univalent map. Moreover, $\pm e_1$ are regular contact points for f. Indeed, $f_1(z, w) = \varphi(z)$ which is smooth at ± 1 and $\varphi(\pm 1) = 1$. Hence

$$\liminf_{r\to 1} \frac{1 - \|f(\pm r e_1)\|}{1 - r} \le \liminf_{r\to 1} \frac{|1 - \varphi(\pm r)|}{1 - r} = |\varphi'(\pm 1)| < +\infty.$$

Also, by construction $f(\pm e_1) = e_1$.

We are left to show that we can find $\delta, \epsilon > 0$ such that

(1) $\Phi(\mathbb{H}^2) \subset \mathbb{H}^2$,

(2) Φ is injective.

(1) Let $(z, w) \in \mathbb{H}^2$. Since $|1 + \phi(z)| \ge 1 + \mathsf{Re}\,\phi(z) \ge 1 + 3 = 4$ and $|z|/(|1 + z|) \le 1$,

$$\frac{\sqrt{\alpha}\delta|w|}{|1 + \phi(z)|} \le \frac{\sqrt{\alpha}\delta|w|}{4}, \qquad \frac{\epsilon|z|}{|1 + z|} \le \epsilon.$$

Let $\epsilon > 0$ be such that $\epsilon^2 < 3/4$, and let $\delta > 0$ be such that $\delta < 2$. Since $3 < \mathsf{Re}\,\phi(z)$ and $\mathsf{Re}\,\phi(z) \ge \alpha \mathsf{Re}\,z$,

$$\begin{aligned}\left|\frac{\sqrt{\alpha}\delta w}{1 + \phi(z)} + \frac{\epsilon z}{1 + z}\right|^2 &\le 2\left(\frac{\alpha\delta^2|w|^2}{|1 + \phi(z)|^2} + \frac{\epsilon^2|z|^2}{|1 + z|^2}\right) \le 2\left(\frac{\alpha\delta^2|w|^2}{(1 + 3)^2} + \epsilon^2\right)\\ &\le \frac{\alpha|w|^2}{2} + \frac{3}{2} \le \frac{\alpha|w|^2}{2} + \frac{\mathsf{Re}\,\phi(z)}{2} \le \frac{\alpha\mathsf{Re}\,z}{2} + \frac{\mathsf{Re}\,\phi(z)}{2} \le \mathsf{Re}\,\phi(z),\end{aligned}$$

proving that $\Phi(z, w) \in \mathbb{H}^2$.

(2) Suppose $\Phi(z_0, w_0) = \Phi(z_1, w_1)$. If $z_0 = z_1$ then clearly $w_0 = w_1$, so assume $z_0 \neq z_1$. Then $a := \phi(z_0) = \phi(z_1)$ which, up to relabeling, implies $z_0 \in U$ and $z_1 \in V$. That is, $\|w_0\|^2 \le \mathsf{Re}\,z_0 \le |z_0| < R_0$ and $|z_1| > R_\infty$. Thus

$$\frac{\sqrt{\alpha}\delta w_0}{1 + a} + \frac{\epsilon z_0}{1 + z_0} = \frac{\sqrt{\alpha}\delta w_1}{1 + a} + \frac{\epsilon z_1}{1 + z_1}.$$

Note that for all $(z, w) \in \mathbb{H}^2$,

$$\left|\frac{\sqrt{\alpha}\delta w}{1 + \phi(z)}\right|^2 = \frac{\alpha\delta^2|w|^2}{|1 + \phi(z)|^2} \le \frac{\delta^2\alpha\mathsf{Re}\,z}{|1 + \phi(z)|^2} \le \frac{\delta^2\mathsf{Re}\,\phi(z)}{|1 + \phi(z))|^2} \le \delta^2.$$

Hence,

$$\frac{\epsilon R_\infty}{1+R_\infty}-\delta \leq \left|\frac{\epsilon z_1}{1+z_1}\right|-\left|\frac{\sqrt{\alpha}\delta w_1}{1+a}\right| \leq \left|\frac{\sqrt{\alpha}\delta w_1}{1+a}+\frac{\epsilon z_1}{1+z_1}\right| = \left|\frac{\sqrt{\alpha}\delta w_0}{1+a}+\frac{\epsilon z_0}{1+z_0}\right| \leq \delta+\frac{\epsilon R_0}{1-R_0}.$$

Therefore,

$$\epsilon\left(\frac{R_\infty}{1+R_\infty}-\frac{R_0}{1-R_0}\right) \leq 2\delta.$$

But $\frac{R_\infty}{1+R_\infty}-\frac{R_0}{1-R_0} = 1-r_1/2-r_{-1}/(2-2r_{-1}) > 1-1/2-1/2 > 0$, hence, if δ is sufficiently small the previous condition is never satisfied and the map is univalent.

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